EQUATIONS AND VARIABLES ASSOCIATED WITH THE LINEAR DIFFERENTIAL EQUATION.

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My attention was first drawn to this subject by reading a memoir by A. R. Forsyth, published in the Phil. Trans. of the Royal Society, Vol. 179 (1888), pp. 377–489, and entitled Invariants, Covariants, and Quotient-derivatives associated with Linear Differential Equations.

Also, reference may be made to Professor Craig's Treatise, Chapter XIII; or my dissertation, entitled Invariants and Equations associated with the Linear Differential Equation.

As much is gained through the adoption of a convenient and brief form of notation, I introduce the following:

$$\frac{dy}{dx} \equiv y', \quad \frac{d^n y}{dx^n} \equiv y^{(n)}, \quad \text{etc.},$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv (y_1 y_2') \equiv (1\ 2'),$$

$$\begin{vmatrix} y_1, & y_2, & y_3, & \ldots, & y_k, & \ldots, & y_n \\ y_1', & y_2', & y_3', & \ldots, & y_k', & \ldots, & y_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & y_3^{(n-1)}, & \ldots, & y_k^{(n-1)}, & \ldots, & y_n^{(n-1)} \end{vmatrix} \equiv (y_1 y_2' y_3'' \ldots y_k^{(k-1)} \ldots y_n^{n-1}) \equiv (1\ 2'\ 3'' \ldots k^{(k-1)} \ldots n^{(n-1)}).$$

I.—Associate Equations.

A linear differential equation of the nth order

$$A = 0 = y^{(n)} + \frac{n\,.\,(n-1)}{1\,.\,2} P_2 y^{(n-2)} + \ldots + \frac{n\,!}{r\,!\,(n-)\,r\,!} P_r y^{(n-r)} + \ldots + P_n y$$

has n fundamental solutions $y_1, y_2, \ldots, y_n$. From any two of these y_α, y_β we can form the function $(y_\alpha y_\beta' - y_\alpha' y_\beta)$, or in the abbreviated notation $(\alpha\beta')$, and from the n y's $\frac{n(n-1)}{2\,!}$ such functions, in general independent, which may be taken as the solutions of a linear differential equation of the $\frac{n(n-1)}{1\,.\,2}$th order

$$A_1 = 0\,.$$

Taking any three of the solutions y, we can form functions $(y_\alpha y_\beta' y_\gamma'')$ or $(\alpha\beta'\gamma'')$, $\frac{n(n-1)(n-2)}{3!}$ in number, in general independent, and thus forming the solutions of a linear differential equation of the $\frac{n(n-1)(n-2)}{3!}$ th order

$$A_2 = 0\,.$$

Similarly, from the combination of four solutions y we obtain $\frac{n!}{4!\,(n-4)!}$ functions $(\alpha\beta'\gamma''\delta''')$ which are the solutions of a linear differential equation

$$A_3 = 0\,.$$

Proceeding thus, and combining the y's five at a time, we obtain the solutions of $A_4 = 0$ of order $\frac{n!}{5!\,(n-5)!}$; combining them six at a time, we form the solutions of $A_5 = 0$ of order $\frac{n!}{6!\,(n-6)!}$; finally, combining the y's $n-1$ at a time, n functions are obtained solutions of a linear differential equation of the nth order

$$A_{n-2} = 0\,.$$

These equations $A_1 = 0$, $A_2 = 0$, $A_3 = 0, \ldots, A_{n-2} = 0$ are called, respectively, the first, second, third, ..., $(n-2)$th associate equations of the original equation $A = 0$.

The last, viz, $A_{n-2} = 0$ has long been known as the adjoint of $A = 0$, or Lagrange's adjoint equation. Let $u_1, u_2, \ldots, u_n$, denote its solutions, where

$$u_\alpha \equiv (-1)^{\alpha-1} \begin{vmatrix} y_1, & y_2, & \ldots, & y_{\alpha-1}, & y_{\alpha+1}, & \ldots, & y_n \\ y'_1, & y'_2, & \ldots, & y'_{\alpha-1}, & y'_{\alpha+1}, & \ldots, & y'_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & \ldots, & y_{\alpha-1}^{(n-2)}, & y_{\alpha+1}^{(n-2)}, & \ldots, & y_n^{(n-2)} \end{vmatrix}$$

$$\equiv (-1)^{\alpha-1}(y_1 y'_2 \cdots y_{\alpha-1}^{(\alpha-2)}\, y_{\alpha+1}^{(\alpha-1)} \cdots y_n^{(n-2)}).$$

When we consider the determinant

$$\begin{vmatrix} y_1, & y_2, & y_3, & \ldots, & y_n \\ y'_1, & y'_2 & y'_3, & \ldots, & y'_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & y_3^{(n-2)}, & \ldots, & y_n^{n-2} \\ y_1^{(l)}, & y_2^{(l)}, & y_3^{(l)}, & \ldots, & y_n^{(l)} \end{vmatrix}$$

we see that it vanishes for all values of l from 0 to $n-2$, and as the first $n-1$ rows give the solutions of the adjoint $A_{n-2}=0$, we have

$$y_1^{(l)}u_1 + y_2^{(l)}u_2 + y_3^{(l)}u_3 + \ldots + y_n^{(l)}u_n = 0\,, \quad (l=0,\,1,\,\ldots,\,n-2)\,. \qquad (1)$$

The determinant also vanishes when we replace the u's by their derivatives $u_1^{(k)}\ u_2^{(k)}\ \ldots\ u_n^{(k)}$ provided $k+l \gtrless n-1$. Thus we have the more general relation

$$y_1^{(k)}u_1^{(l)} + y_2^{(k)}u_2^{(l)} + \ldots + y_n^{(k)}u_n^{(l)} = 0, \; (k=0,1,\ldots,n-1;\, l=0,1,\ldots,n-1) \quad (2)$$

provided $k+l \gtrless n-1$.

These relations between the solutions of an equation and its adjoint become more interesting when the equation is self-adjoint, i. e. when $A_{n-2}=0$ is the same equation as $A=0$.

In the year 1889 I pointed out to Professor Craig and Professor Forsyth the following relation between the associate equations: The kth associate of the adjoint equation is the same as the $(n-2-k)$th associate of the original equation. This is proven in my dissertation and is included in a theorem due to Clebsch, published in the "Abhandlungen der Kön. Gesellschaft der Wissenschaften zu Göttingen, Band XVII," Ueber die Fundamentalaufgabe der Invarianttheorie.

II.—The Self-Adjoint Equation.

When $A_{n-2}=0$ is the same equation as $A=0$ the u must be linear functions of the y, thus:

$$u_i = \sum_1^n a_{ik} y_k\,, \qquad (i=1,2,\ldots,n) \quad (3)$$

where the a_{ik} are constants.

The condition necessary and sufficient that $A=0$ should be self-adjoint is that the invariants with odd suffix vanish. This theorem found in an article by Professor Brioschi published 1891 was proven by myself two years earlier (see Professor Craig's Treatise, p. 495).

If in (1) we let $l=0$ and then substitute the u from (3) we obtain

$$\varphi_{(y)} \equiv \sum_1^n \sum_1^n a_{ik} y_i y_k = 0\,.$$

To consider more closely the constants a_{ki}, let $n=3$; then $A=0$ becomes

$$y''' + 3P_2 y' + P_3 y = 0\,,$$

where

$$3P_2 = -\frac{(2\ 3''')}{(2\ 3')} \text{ and } P_3 = \frac{(2'\ 3''')}{(2\ 3')}\,,$$

and

$$\varphi(y) = a_1 y_1^2 + (a_{12} + a_{21})\, y_1 y_2 + (a_{13} + a_{31})\, y_1 y_3 + a_{22} y_2^2 + (a_{23} + a_{32})\, y_2 y_3 + a_{33} y_3^2 = 0 .$$

From (3) we obtain

$$\begin{aligned} u_1 &= (y_2 y_3') = a_{11} y_1 + a_{12} y_2 + a_{13} y_3 , \\ u_2 &= (y_3 y_1') = a_{21} y_1 + a_{22} y_2 + a_{23} y_3 , \\ u_3 &= (y_1 y_2') = a_{31} y_1 + a_{32} y_2 + a_{33} y_3 . \end{aligned}$$

From u_1 and its first and second derivatives we obtain a_{11}, a_{12} and a_{13} by solving the equations

$$\begin{aligned} u_1 &= (y_2 y_3') = a_{11} y_1 + a_{12} y_2 + a_{13} y_3 , \\ u_1' &= (y_2 y_3'') = a_{11} y_1' + a_{12} y_2' + a_{13} y_3' , \\ u_1'' &= (y_2' y_3'') - 3P_2\,(y_2 y_3') = a_{11} y_1'' + a_{12} y_2'' + a_{13} y_3'' ; \end{aligned}$$

viz.

$$\begin{aligned} (1\ 2'\ 3'')\, a_{11} &= 2\,(2'\ 3'')\,(2\ 3') - 3\,(2\ 3')^2 P_2 - (2\ 3'')^2 , \\ (1\ 2'\ 3'')\, a_{12} &= (2\ 3')\,(3'\ 1'') + (2'\ 3'')\,(3\ 1') - (2\ 3'')\,(3\ 1'') - 3P_2\,(2\ 3')\,(3\ 1') , \\ (1\ 2'\ 3'')\, a_{13} &= (2\ 3')\,(1'\ 2'') + (2'\ 3'')\,(1\ 2') - (2\ 3'')\,(1\ 2'') - 3P_2\,(2\ 3')\,(1\ 2') . \end{aligned}$$

As $(1\ 2'\ 3'')$ is constant, and also a_{11}, a_{12}, a_{13}, so must the right members of these equations be constant. If in these equations we interchange the suffixes the right members remain unchanged in value ; therefore

$$(1\ 2'\ 3'')\, a_{12} = (1\ 2'\ 3'')\, a_{21} , \quad a_{13} = a_{31} , \quad a_{23} = a_{32} .$$

Proceeding in a similar way for $n = 4$, we obtain

$$\begin{aligned} a_{11} &= 4u_1^2 \theta_3 , & a_{12} &= (4'\ 3'') - (4\ 3''') - 6P_2(4\ 3') , \\ a_{13} &= (2'\ 4'') - (2\ 4''') - 6P_2(2\ 4') , & a_{14} &= (3'\ 2'') - (3\ 2''') - 6P_2(3\ 2') . \end{aligned}$$

Thus $a_{11} = 0$, as the invariant θ_3 is supposed to vanish.

Had we solved for a_{21}, a_{31}, and a_{41}, we should have found that $a_{21} = -a_{12}$, $a_{31} = -a_{13}$, and $a_{41} = -a_{14}$. That these results hold for all values of n will be seen by the following considerations :—

$$\frac{d\varphi}{dx} = \frac{\partial \varphi}{\partial y_1} y_1' + \frac{\partial \varphi}{\partial y_2} y_2' + \ldots + \frac{\partial \varphi}{\partial y_n} y_n' = 0 ,$$

$$\frac{d^2\varphi}{dx^2} = \frac{\partial \varphi}{\partial y_1} y_1'' + \frac{\partial \varphi}{\partial y_2} y_2'' + \ldots + \frac{\partial \varphi}{\partial y_n} y_n'' + 2\varphi(y') = 0 ;$$

but $\varphi(y')$ by equations (2) $= 0$.

Similarly we see that

$$\sum_1^n{}_i \frac{\partial\varphi}{\partial y_i} y_1^{(r)} = 0\,, \qquad (r = 0, \dots, n-2) \qquad (4)$$

which equations are similar to (1); and as the determinant formed with the y and their derivatives is not zero, $\frac{\partial\varphi}{\partial y_i}$ must be proportional to u_i; i. e.

$$\frac{\partial\varphi}{\partial y} = ku_i\,. \qquad (i = 1, 2, \dots, n)$$

Then it follows that

$$2a_{11}y_1 + (a_{12} + a_{21})y_2 + (a_{31} + a_{13})y_3 + \dots$$
$$+ (a_{1n} + a_{n1})y_n \equiv k(a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \dots + a_{1n}y_n)\,,$$
$$(a_{21} + a_{12})y_1 + 2a_{22}y_2 + (a_{23} + a_{32})y_3 + \dots$$
$$+ (a_{2n} + a_{n2})y_n \equiv k(a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \dots + a_{2n}y_n)\,,$$

. . . .

As all the a_{ik}'s cannot vanish we must have $k = 0$ or 2. For $k = 0$,

$$a_{11} = a_{22} = a_{33} = a_{44} = \quad = a_{nn} = 0$$

and

$$a_{ik} = -a_{ki}\,. \qquad (i = 1, 2, \dots, n\,;\ k = 1, 2, \dots, n)$$

For $k = 2$,

$$a_{ik} = a_{ki}\,.$$

In order that the determinant of the a's may not vanish n must be even when $a_{ik} = -a_{ki}$. Thus we can announce the theorems,

When n is odd, the determinant of the a is symmetric.

When n is even, the determinant of the a is skew symmetric and the equation $\varphi(y)$ vanishes identically.

We then have to do with a linear complex, included in

$$u_1y_1^{(l)} + u_2y_2^{(l)} + \dots + u_ny_n^{(l)} = 0\,. \qquad (l = 1, 2, \dots n-2)\,,$$

or

$$a_{12}(1\,2^{(l)}) + a_{13}(1\,3_1^{(l)}) + \dots + a_{1n}(1n^{(l)}) + a_{23}(2\,3^{(l)}) + \dots$$
$$+ a_{ik}(ik^{(l)}) + \dots + a_{n-1,n}(n-1, n^{(l)}) = 0\,. \quad (5)$$

Thus we see that the first associate of a self-adjoint equation is of order $\overline{\overline{<}}\, \frac{1}{2}n(n-1) - 1$.

When we take into consideration the relations existing between the associate variables, the application of the possible groups of substitutions shows that some of the coefficients a_{ik} may vanish.

By the introduction of a linear substitution we can cause all the a_{ik} to vanish except those for which $i + k = n + 1$ and that lie in the diagonal which is not principal, and these may have the value ± 1. For $n = 4$ (5) becomes

$$(y_1y_4') + (y_2y_3') = 0\,, \quad (y_1y_n'') + (y_2y_3'') = 0\,.$$

Taking $A \equiv y^{iv} + 6P_2y'' + 4P_3y' + P_4y = 0$, where $P_3 = \frac{3}{2} P_2'$, the solutions of the first associate will be

$$v_1 - (1\ 2')\,,\ v_2 = (1\ 3')\,,\ v_3 = (1\ 4')\,,\ v_4 = (2\ 4')\,,\ v_5 = (3\ 4')\,.$$

Usually it is difficult to form the first associate, but the use of the values found for a_{ik} simplifies the process.

$$v = (1\ 2')\,,$$

$$v^{\text{i}} = (1\ 2'')\,,$$

$$v^{\text{ii}} = 2(1'\ 2'') - 6P_2v + a_{34}\,,$$

$$v^{\text{iii}} + 6P_2v' + 6P_2'v = 2(1'\ 2''')\,,$$

$$v^{\text{iv}} + 6P_2v'' + 12P_2'v + 6P_2''v = 2\,[(1''\ 2''') + P_4v - 3P_2(v'' + 6P_2u - a_{34})]\,,$$

$$A_1 \equiv v^{\text{v}} - 12\,P_2v''' + 18P_2'v'' + (18P_2'' + 36P_2^2 - 4P_4)v'$$
$$+ (6P_2''' + 36P_2P_2' - 2P_4')\,v = 0\,.$$

Forming the invariants θ_3 and θ_5 for this, we find that they vanish; A_1 is, therefore, self-adjoint.

It is not difficult to show that $A_1 = 0$ has for its first associate an equation of which the solutions are

$$y_1^2,\ y_1y_2,\ y_1y_3,\ y_1y_4,\ y_2^2,\ y_2y_3,\ y_2y_4,\ y_3^2,\ y_3y_4,\ y_4^2,\ \ldots\,.$$

After the substitution we have,

$$u_a \equiv (-1)^{a-1}(y_1y_2' \ldots y_{a-1}^{(a-2)}\, y_{a+1}^{(a-1)} \ldots y_n^{(n-2)}) = a_{ak}y_k\,,$$

where $k = n + 1 - a$. If in any portion of the plane the y's have the form x^m, the exponents belonging to y_k being m_k, we obtain the following relations:

For $n = 3$,

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = a_{31}y_1 \text{ and } m_1 + m_2 - 1 = m_1\,;\ \text{whence } m_2 = 1, \text{ and } m_1 + m_3 = 2\,.$$

For $n = 6$, we have

$(y_1 y_2^{\text{i}} y_3^{\text{ii}} y_4^{\text{iii}} y_5^{\text{iv}}) = a_k y_1$, then $m_1 + m_2 + m_3 + m_4 + m_5 - 10 = m_1$;

$(y_1 y_2^{\text{i}} y_3^{\text{ii}} y_4^{\text{iii}} y_6^{\text{iv}}) = c y_2$, and $m_1 + m_2 + m_3 + m_4 + m_6 - 10 = m_2$;

similarly, $m_1 + m_2 + m_3 + m_5 + m_6 - 10 = m_3$;

$$m_1 + m_2 + m_4 + m_5 + m_6 - 10 = m_4 .$$

Adding the first three and subtracting the fourth twice gives $2(m_3 + m_4) = 10$, and similarly for the rest; so that

$$m_1 + m_6 = m_2 + m_5 = m_3 + m_4 = 5 ,$$

and in like manner

$$m_1 + m_n = m_2 + m_{n-1} = m_3 + m_{n-2} = \;..= m_s + m_{n-s+1} = n - 1 .$$

Where y_k be of the form $e^{m_k x}$, the same relations exist between the m, but $m_s + m_{n+1-s} = 0 \quad (s = 1, 2, 3, 4, \ldots)$.

In an interesting and exhaustive article in Crelle, Bd. 113, G. Wallenberg treats the class of equations known as Fuchsian equations. They have rational coefficients, the integrals behave regularly in the neighborhood of the singular points, and the roots of the indicial (*determinirende*, fundamental) equation are all rational numbers. Halphen has shown that an equation may be transformed into another with constant coefficients if the absolute invariants are constant.

Then for a self-adjoint equation having its absolute invariants constant and belonging to Fuchs' class. Mr. Wallenberg finds the y in the form

$$y_k = [R_{(x)}]^{-\frac{n-1}{4}} e^{-\gamma\mu_k \int \sqrt{Rx}dx} , \qquad y_{n+1-k} = [R_{(x)}]^{-\frac{n-1}{4}} e^{\gamma\mu_k \int \sqrt{Rx}dx}$$

$$k = 1, 2, 3, \ldots .$$

Then there exists $n - 2$ homogeneous relations of the second order between the y; viz.:

$$y_1 y_n = y_2 y_{n-1} = y_3 y_{n-2} = \ldots = y_k y_{n+1-k} ,$$

$$\left[\frac{y_1}{y_n}\right]^{\mu_k} = \left[\frac{y_k}{y_{n+1-k}}\right]^{\mu_1} . \qquad (k = 1, 2, 3, \ldots)$$

In this $R(x)$ is a rational function which becomes unity so that

$$y_k = \frac{1}{y_{n+1-k}} = e^{-r_k x} .$$

Then we have

$$u_a = (-1)^{a-1}(y_1 y_2' \dots y_{a-1}^{(a-2)} y_{a+1}^{(a-1)} \dots y_n^{(n-2)}).$$

Substituting the values of the y and taking $n = 6$,

$$u_2 = (-2)^{\frac{n}{2}-1}(-1)^{a-1}\frac{r_1 r_2 r_3}{r_2}\begin{vmatrix} 1 & 1 \\ r_1^2 & r_3^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ r_1^2 & r_2^2 & r_3^2 \\ r_1^4 & r_2^4 & r_3^4 \end{vmatrix} y_5 .$$

For $n = 7$,

$$u_2 = (-2)^3(-1)\frac{r_1^3 r_2^3 r_3^3}{r_2^2}\begin{vmatrix} 1 & 1 \\ r_1^2 & r_3^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ r_1^2 & r_2^3 & r_3^2 \\ r_1^4 & r_2^4 & r_3^4 \end{vmatrix} y_6 ;$$

with similar expressions for the other adjoint variables. It is not difficult to find them for the equation of the nth order. They serve to verify the preceding theory.

When all the invariants vanish we have the case in which $A = 0$ may be reduced to the form $y^n = 0$, so that the solutions are

$$y_k = x^{k-1}; \qquad (k = 1, 2, 3, \dots, n)$$

and we have the parabolic relations

$$y_k^2 = y_{k-1}\, y_{k+1} \qquad (k = 1, 2, \dots, n-2)$$